Linear systems – Resit exam (solutions)

Resit exam 2018–2019, Tuesday 9 July 2019, 9:00 – 12:00

Problem 1 (10 points)

To solve the initial value problem

$$\dot{x}(t) + \frac{t}{1+t^2}x(t) = (1+2t^2)\sqrt{1+t^2}, \qquad x(0) = 2, \tag{1}$$

consider the differential equation in the standard form

$$\dot{x}(t) = -\frac{t}{1+t^2}x(t) + (1+2t^2)\sqrt{1+t^2}. \tag{2}$$

Then, a direct computation gives

$$F(t) = -\int \frac{t}{1+t^2}\,\mathrm{d}t = -\tfrac{1}{2}\int \frac{1}{1+u}\,\mathrm{d}u = -\tfrac{1}{2}\ln|1+u| = -\tfrac{1}{2}\ln|1+t^2|, \tag{3}$$

where the substitution $u = t^2$ is used. After simplifying the result as

$$F(t) = -\tfrac{1}{2}\ln|1+t^2| = -\tfrac{1}{2}\ln(1+t^2) = -\ln\left((1+t^2)^{\frac{1}{2}}\right) = -\ln\sqrt{1+t^2}, \tag{4}$$

the integrating factor reads

$$e^{-F(t)} = e^{\ln\sqrt{1+t^2}} = \sqrt{1+t^2}. \tag{5}$$

Now, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sqrt{1+t^2}\,x(t)\right\} &= \sqrt{1+t^2}\,\dot{x}(t) + \frac{t}{\sqrt{1+t^2}}x(t)\\ &= \sqrt{1+t^2}\left(\dot{x}(t) + \frac{t}{1+t^2}x(t)\right)\\ &= (1+t^2)(1+2t^2) = 1+3t^2+2t^4,\end{aligned} \tag{6}$$

where the latter result follows from the substitution of the dynamics (2). By direct integration, we obtain

$$\sqrt{1+t^2}\,x(t) = \int 1+3t^2+2t^4\,\mathrm{d}t = t+t^3+\tfrac{2}{5}t^5+C \tag{7}$$

for some constant C, such that the general solution of (2) reads

$$x(t) = \frac{t+t^3+\frac{2}{5}t^5+C}{\sqrt{1+t^2}}. \tag{8}$$

For $t = 0$, this gives

$$x(0) = C, \tag{9}$$

such that the initial condition (1) leads to

$$C = 2. \tag{10}$$

Problem 2 (4 + 8 = 12 points)

Consider the nonlinear system

$$\dot{x}_1(t) = x_1(t) - x_1(t)x_2(t),$$
$$\dot{x}_2(t) = -x_1^3(t) + u(t).$$

(a) We will compute the equilibrium point $\bar{x} = [\,\bar{x}_1\ \bar{x}_2\,]^{\mathrm{T}}$ for $\bar{u} = 1$. This requires solving the set of equations

$$0 = \bar{x}_1 - \bar{x}_1\bar{x}_2, \tag{11}$$
$$0 = -\bar{x}_1^3 + \bar{u} \tag{12}$$

for $\bar{u} = 1$. From (12), we immediately obtain (recall that only real solutions are considered)

$$\bar{x}_1 = 1, \tag{13}$$

as the unique solutions. Then, the substitution of (13) in (11) leads to $0 = 1 - \bar{x}_2$, such that

$$\bar{x}_2 = 1, \tag{14}$$

is the unique solution. Hence, the unique equilibrium corresponding to $\bar{u} = 1$ reads $\bar{x} = [\,1\ 1\,]^{\mathrm{T}}$.

(b) Before computing the linearized system, introduce the notation

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad f(x,u) = \begin{bmatrix} x_1 - x_1x_2 \\ -x_1^3 + u \end{bmatrix}. \tag{15}$$

Now, define the perturbations from the equilibrium as

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}, \tag{16}$$

such that we have

$$\dot{\tilde{x}} = f(\bar{x} + \tilde{x}, \bar{u} + \tilde{u}). \tag{17}$$

This leads to the linearized dynamics (by the Taylor expansion)

$$\dot{\tilde{x}} = \frac{\partial f}{\partial x}(\bar{x},\bar{u})\tilde{x} + \frac{\partial f}{\partial u}(\bar{x},\bar{u})\tilde{u}. \tag{18}$$

Then, computation of the Jacobian of f with respect to x gives

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} 1 - x_2 & -x_1 \\ -3x_1^2 & 0 \end{bmatrix}, \tag{19}$$

leading to

$$\frac{\partial f}{\partial x}(\bar{x},\bar{u}) = \begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix}. \tag{20}$$

Similarly, we obtain

$$\frac{\partial f}{\partial u}(\bar{x},\bar{u}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{21}$$

Finally, the substitution of the results (20) and (21) in (18) gives

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix}\tilde{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\tilde{u}(t). \tag{22}$$

Problem 3 (4 + 10 + 6 = 20 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

with state $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -7 & -3 \\ 22 & 10 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

(a) A direct computation leads to

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \tag{23}$$

from which one can conclude that

$$\operatorname{rank} \begin{bmatrix} B & AB \end{bmatrix} = 2. \tag{24}$$

Hence, the system is controllable.

(b) As a first step in finding the matrix T, compute

$$\begin{aligned} \Delta_A(\lambda) = \det(\lambda I - A) &= \begin{vmatrix} \lambda + 7 & 3 \\ -22 & \lambda - 10 \end{vmatrix} \\ &= (\lambda + 7)(\lambda - 10) + 66 = \lambda^2 - 3\lambda - 70 + 66 = \lambda^2 - 3\lambda - 4. \end{aligned} \tag{25}$$

Define

$$a_1 = -3, \qquad a_0 = -4, \tag{26}$$

such that $\Delta_A(s) = s^2 + a_1 s + a_0$.

Next, define the vectors

$$q_2 = B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \tag{27}$$

$$q_1 = AB + a_1 B = \begin{bmatrix} -2 \\ 8 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{28}$$

where the results (23) and (26) are used. This leads to the definition of T through its inverse as

$$T^{-1} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \tag{29}$$

which is guaranteed to satisfy

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{30}$$

As a result, the desired values of α_1 and α_2 read

$$\alpha_1 = -a_0 = 4, \qquad \alpha_2 = -a_1 = 3. \tag{31}$$

For completeness, we give T as

$$T = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}. \tag{32}$$

(c) To find the desired feedback, note that

$$\Delta_{A+BF}(s) = \Delta_{T(A+BF)T^{-1}}(s), \tag{33}$$

and

$$T(A+BF)T^{-1} = TAT^{-1} + TBFT^{-1}. \tag{34}$$

After denoting

$$FT^{-1} = \begin{bmatrix} f_0 & f_1 \end{bmatrix}, \tag{35}$$

the result (30) gives

$$TAT^{-1} + TBFT^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_0 & f_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f_0 - a_0 & f_1 - a_1 \end{bmatrix}, \tag{36}$$

such that

$$\Delta_{A+BF}(s) = s^2 + (a_1 - f_1)s + (a_0 - f_0). \tag{37}$$

As we would like to place the eigenvalues of $A + BF$ at -3 and -2, consider the desired characteristic polynomial

$$p(s) = (s+3)(s+2) = s^2 + 5s + 6. \tag{38}$$

Equating the polynomials (37) and (38) leads to

$$f_0 = a_0 - 6 = -4 - 6 = -10, \tag{39}$$
$$f_1 = a_1 - 5 = -3 - 5 = -8, \tag{40}$$

after which the desired feedback matrix F can be found by solving the linear system

$$FT^{-1} = F \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -10 & -8 \end{bmatrix}. \tag{41}$$

This leads to

$$F = \begin{bmatrix} -19 & -9 \end{bmatrix}. \tag{42}$$

Problem 4 (14 points)

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -a & -2 & -a \end{bmatrix} x(t) \tag{43}$$

and denote

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -a & -2 & -a \end{bmatrix}. \tag{44}$$

The system is asymptotically stable if and only if $\sigma(A) \subset \mathbb{C}_-$, which can equivalently checked by determining stability of the characteristic polynomial of A, i.e., Δ_A. As A is in so-called companion form, it immediately follows that

$$\Delta_A(s) = s^4 + as^3 + 2s^2 + as + b. \tag{45}$$

Stability of the polynomial Δ_A can be verified using the Routh-Hurwitz test, leading to the following table:

	s^4	s^3	s^2	s^1	s^0	
$a\times$	1	a	2	a	b	Δ_A
$1\times$	a		a			
		a^2	a	a^2	ab	result of step 1
$1\times$		a	1	a	b	after division by a: Δ'_A
$a\times$		1		b		
			1	$a(1-b)$	b	result of step 2: Δ''_A

A necessary condition for stability of Δ_A is that all coefficients have the same sign, which implies that

$$a > 0, \qquad b > 0 \tag{46}$$

is a necessary condition.

After step 1 (note that division by a is allowed as we assume the necessary condition (46)), no new conditions are found.

However, applying the same reasoning to the polynomial Δ''_A obtained at step 2, we obtain

$$a > 0, \qquad 0 < b < 1 \tag{47}$$

as a necessary condition for stability of Δ''_A. However, as the polynomial Δ''_A is quadratic, we know that (47) is also sufficient for stability.

Hence, by the Routh-Hurwitz criterion, a necessary and sufficient condition for stability of Δ_A (and, hence, asymptotic stability of the linear system) is

$$a > 0, \qquad 0 < b < 1. \tag{48}$$

Problem 5 (8 + 4 + 4 = 16 points)

Consider the matrices

$$A = \begin{bmatrix} -1 & -1 \\ -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}. \tag{49}$$

(a) As a first step in computing the matrix exponential, an eigenvalue decomposition of A will be considered. Therefore, the characteristic polynomial is computed as

$$\Delta_A(s) = \det(sI - A) = \begin{vmatrix} s+1 & 1 \\ 2 & s \end{vmatrix} = (s+1)s - 2 = s^2 + s - 2 = (s+2)(s-1) \tag{50}$$

such that A has the eigenvalues

$$\lambda_1 = -2, \qquad \lambda_2 = 1. \tag{51}$$

The corresponding eigenvalues can be found as

$$0 = (\lambda_1 I - A)v_1 = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} v_1 \qquad \Longrightarrow \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{52}$$

$$0 = (\lambda_2 I - A)v_2 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} v_2 \qquad \Longrightarrow \qquad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \tag{53}$$

After defining

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \tag{54}$$

we can write

$$A = T\Lambda T^{-1}, \tag{55}$$

where

$$T^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}. \tag{56}$$

Now, the matrix exponential can be found as

$$\begin{aligned} e^{At} = e^{T\Lambda T^{-1}t} &= Te^{\Lambda t}T^{-1} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2e^{-2t} & e^{-2t} \\ e^{t} & -e^{t} \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2e^{-2t} + e^{t} & e^{-2t} - e^{t} \\ 2e^{-2t} - 2e^{t} & e^{-2t} + 2e^{t} \end{bmatrix}. \end{aligned} \tag{57}$$

Next, consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

with $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and the matrices A, B, C given by (49).

(b) Recall that the system is externally stable if

$$\lim_{t\to\infty} Ce^{At}B = 0. \tag{58}$$

A direct computation yields

$$\begin{aligned} Ce^{At}B &= \frac{1}{3}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2e^{-2t}+e^{t} & e^{-2t}-e^{t} \\ 2e^{-2t}-2e^{t} & e^{-2t}+2e^{t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3e^{-2t} \\ 3e^{-2t} \end{bmatrix} \\ &= 3e^{-2t}. \end{aligned} \tag{59}$$

It is clear that

$$\lim_{t\to\infty} 3e^{-2t} = 0, \tag{60}$$

such that the system is externally stable.

(c) The transfer function can be found as the Laplace transform of $Ce^{At}B$ (as $D = 0$), which leads to

$$\begin{aligned} T(s) = \int_0^\infty 3e^{-2t}e^{-st}\,\mathrm{d}t &= 3\int_0^\infty e^{-(s+2)t}\,\mathrm{d}t \\ &= -\frac{3e^{-(s+2)t}}{s+2}\bigg|_0^\infty \\ &= \frac{3}{s+2}\left(1-\lim_{t\to\infty} e^{-(s+2)t}\right) \\ &= \frac{3}{s+2} \end{aligned} \tag{61}$$

for all $s \in \mathbb{C}$ such that $\Re(s) > -2$. However, as usual, the issue of convergence will be ignored and we simply write

$$T(s) = \frac{3}{s+2}. \tag{62}$$

Problem 6 (8 + 10 = 18 points)

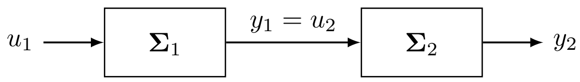

Figure 1. Cascade interconnection of two systems

Consider two linear systems

$$\mathbf{\Sigma}_i : \quad \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad y_i(t) = C_i x_i(t)$$

for $i \in \{1, 2\}$ and their cascade interconnection given by $u_2(t) = y_1(t)$ as given in Figure 1. Then, the dynamics of the interconnection can be described as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_1(t).$$

Assume that A_1 and A_2 have no eigenvalues in common (i.e., $\sigma(A_1) \cap \sigma(A_2) = \emptyset$) and that the interconnection is controllable.

(a) Denote

$$\bar{A} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \tag{63}$$

We have that the pair $(\bar{A}, \bar{B})$ is controllable, i.e., by the Hautus test,

$$\operatorname{rank} \begin{bmatrix} \bar{A} - \lambda I & \bar{B} \end{bmatrix} = n_1 + n_2, \tag{64}$$

for all $\lambda \in \sigma(\bar{A})$, with n_1 and n_2 the state-space dimensions of systems 1 and 2, respectively. An equivalent characterization is given by the implication

$$v^{\mathrm{T}} \begin{bmatrix} \bar{A} - \lambda I & \bar{B} \end{bmatrix} = 0 \qquad \Longrightarrow \qquad v = 0 \tag{65}$$

for all $\lambda \in \sigma(A)$.

Now, after partitioning $v^{\mathrm{T}} = [\, v_1^{\mathrm{T}} \; v_2^{\mathrm{T}} \,]$ and using the definitions (63), we have

$$[\, v_1^{\mathrm{T}} \; v_2^{\mathrm{T}} \,] \begin{bmatrix} A_1 - \lambda I & 0 & B_1 \\ B_2 C_1 & A_2 - \lambda I & 0 \end{bmatrix} \qquad \Longrightarrow \qquad v_1 = 0,\, v_2 = 0 \tag{66}$$

for all $\lambda \in \sigma(\bar{A})$, after which rewriting the left-hand-side leads to

$$\begin{bmatrix} v_1^{\mathrm{T}}(A_1 - \lambda I) + v_2^{\mathrm{T}} B_2 C_1 & v_2^{\mathrm{T}}(A_2 - \lambda I) & v_1^{\mathrm{T}} B_1 \end{bmatrix} = 0 \qquad \Longrightarrow \qquad v_1 = 0,\, v_2 = 0. \tag{67}$$

Now, note that due to the lower block triangular structure of $\bar{A}$ we have that $\sigma(\bar{A}) = \sigma(A_1) \cup \sigma(A_2)$ and take $\lambda \in \sigma(A_1)$. Then, as A_1 and A_2 have no eigenvalues in common, $v_2^{\mathrm{T}}(A_2 - \lambda I) = 0$ implies that $v_2 = 0$ such that the implication (67) reduces to

$$\begin{bmatrix} v_1^{\mathrm{T}}(A_1 - \lambda I) & 0 & v_1^{\mathrm{T}} B_1 \end{bmatrix} = 0 \qquad \Longrightarrow \qquad v_1 = 0 \tag{68}$$

for all $\lambda \in \sigma(A_1)$. This is, by the Hautus test, a necessary and sufficient condition for controllability of the pair (A_1, B_1).

(b) This result can be proven similarly. Now, take $\lambda \in \sigma(A_2)$. In this case, $v_1^{\mathrm{T}}(A_1 - \lambda I) + v_2^{\mathrm{T}} B_2 C_1 = 0$ implies that

$$v_1^{\mathrm{T}} = -v_2^{\mathrm{T}} B_2 C_1 (A_1 - \lambda I)^{-1} = v_2^{\mathrm{T}} B_2 C_1 (\lambda I - A_1)^{-1}. \tag{69}$$

Substitution of this result in (67) leads to

$$\begin{bmatrix} 0 & v_2^{\mathrm{T}}(A_2 - \lambda I) & v_2^{\mathrm{T}} B_2 C_1 (\lambda I - A_1)^{-1} B_1 \end{bmatrix} = 0 \qquad \Longrightarrow \qquad v_2 = 0, \tag{70}$$

which implies that

$$\operatorname{rank} \begin{bmatrix} A_2 - \lambda I & B_2 C_1 (\lambda I - A_1)^{-1} B_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_2 - \lambda I & B_2 T_1(\lambda) \end{bmatrix} = n_2. \tag{71}$$